Subsingular vectors in Verma modules, and tensor product modules over the twisted Heisenberg-Virasoro algebra and $W(2,2)$ algebra

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Abstract

We show that subsingular vectors exist in Verma modules over $W(2,2)$, and present a subquotient structure of these modules. We prove conditions for irreducibility of a tensor product of intermediate series module with the highest weight module. Relations to intertwining operators over vertex operator algebra associated to $W(2,2)$ is discussed.

Also, we study a tensor product of intermediate series and highest weight module over the twisted Heisenberg-Virasoro algebra, and present series of irreducible modules with infinite-dimensional weight spaces.

Keywords: W(2,2) algebra, Heisenber-Virasoro algebra, highest weight module, subsingular vector, intermediate series, vertex algreba, intertwining operator
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1 Introduction

Lie algebra $W(2,2)$ was first introduced by W. Zhang and C. Dong in [ZD] as a part of classification of simple vertex operator algebras generated by two weight two vectors. It is an extension of a well known Virasoro algebra Vir, and its representation theory is somewhat similar to that of Vir. Criterion for irreducibility of Verma modules over $W(2,2)$ was given in [ZD], and the structure of those modules was discussed in [JP]. However, authors in [JP] overlooked an important fact - that a submodule generated by a singular vector is not necessarily isomorphic to some Verma module; and therefore missed an interesting possibility - a subsingular vector in Verma module. In Section 3 we present new results on structure of Verma modules (Theorem 3.1), and formulas for subsingular vector. Necessary condition for existence of a subsingular vector is given, and many examples supporting a conjecture that this condition is sufficient are shown.

It was proved in [LZ], that every irreducible weight module over $W(2,2)$ with finite-dimensional weight subspaces is either the highest or lowest weight module, or a module belonging to an intermediate series. Modules with infinite-dimensional weight subspaces over affine Kac-Moody algebra (see [CP], [Ad1], [Ad2], [Ad3]) and Virasoro algebra ([Zh], [R2], [CGZ]) have been studied recently, motivated by their connection with theory of vertex operator algebras (VOAs) and fusion rules in conformal field theory. Based on modules studied in these papers, we consider a tensor product of an irreducible module from intermediate series, and an irreducible highest weight module. We show in Section 4 how these modules can be obtained from intertwining operators for modules over VOA associated to $W(2,2)$. In Section 5 we classify irreducible tensor products (Theorem 5.1). Existence of, and a formula for a subsingular vector is crucial in this analysis, since generic singular vectors are of no use (Theorem 5.7). We show that a tensor product module contains an irreducible submodule with infinite-dimensional weight subspaces if and only if a subsingular vector exists (Corollary 5.9). Different highest weight modules occur as subquotients in reducible tensor product modules. Some of these subquotients are related to intertwining operators.

In section 7 we discuss irreducibility of tensor product module over the twisted Heisenberg-Virasoro algebra at level zero. This algebra was studied in [B], as it appears in the construction of modules for the toroidal Lie algebras [B2]. We present a rich series of irreducible tensor products.

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2 Lie algebra $W(2,2)$

$W(2,2)$ is a complex Lie algebra with basis $\{W_n, L_n, C, : n \in \mathbb{Z}\}$, and a Lie bracket

$$[L_n, L_m] = (n-m)\, L_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}C, \tag{1}$$

$$[L_n, W_m] = (n-m)\, W_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}C,$$

$$[W_n, W_m] = [\mathcal{L}, C] = 0.$$

In this paper we write $\mathcal{L} = W(2,2)$, a notation used in [LZ]. Obviously, $\{L_n, C, : n \in \mathbb{Z}\}$ spans a copy of Virasoro algebra. Triangular decomposition is given by

$\mathcal{L} = \mathcal{L}_- \oplus \mathcal{L}_0 \oplus \mathcal{L}_+$ where

$$\begin{aligned}
\mathcal{L}_+ &= \bigoplus_{n>0} (\mathbb{C}L_n + \mathbb{C}W_n),\\
\mathcal{L}_- &= \bigoplus_{n>0} (\mathbb{C}L_n + \mathbb{C}W_n),\\
\mathcal{L}_0 &= \mathbb{C}L_0 + \mathbb{C}W_0 + \mathbb{C}C.
\end{aligned}$$

However, W_0 does not act semisimply on the rest of $\mathcal{L}$. Algebra $\mathcal{L}$ is $\mathbb{Z}$-graded by eigenvalues of L_0, namely $\mathcal{L}_n = \mathbb{C}L_{-n} + \mathbb{C}W_{-n} + \delta_{n,0}\mathbb{C}C$.

Let $U(\mathcal{L})$ a universal enveloping algebra, and $\mathcal{I}$ a left ideal generated by $\{L_n, W_n, C - c\mathbf{1}, L_0 - h\mathbf{1}, W_0 - h_W\mathbf{1} : n \in \mathbb{N}\}$ for fixed $c, h, h_W \in \mathbb{C}$. Then $V(c, h, h_W) := U(\mathcal{L})/\mathcal{I}$ is Verma module with central charge c, and highest weight (h, h_W) or, simply, with highest weight (c, h, h_W). It is a free $U(\mathcal{L})$-module, generated by the highest weight vector $v := \mathbf{1} + \mathcal{I}$, and a standard PBW basis

$$\{W_{-m_s} \cdots W_{-m_1} L_{-n_t} \cdots L_{-n_1} v : m_s \geq \cdots \geq m_1 \geq 1, n_t \geq \cdots \geq n_1 \geq 1\}.$$

Throughout the rest of this paper, we assume that a monomial

$$W_{-m_s} \cdots W_{-m_1} L_{-n_t} \cdots L_{-n_1} v$$

is given in PBW basis, unless otherwise noted.

$V(c, h, h_W)$ admits a natural gradation by L_0-eigenspaces, i.e. weight subspaces: $V(c, h, h_W) = \bigoplus_{k\geq 0} V(c, h, h_W)_k$. PBW vectors such that $\sum m_i + \sum n_j = k$ form a basis for $V(c, h, h_W)_k$. Module $V(c, h, h_W)$ has a unique maximal submodule $J(c, h, h_W)$, and $L(c, h, h_W) = V(c, h, h_W)/J(c, h, h_W)$ is unique, up to isomorphism, irreducible highest weight module with highest weight (c, h, h_W).

Theorem 2.1 ([ZD]) *Verma module $V(c, h, h_W)$ is irreducible if and only if $2h_W + \frac{m^2-1}{12}c \neq 0$ for any $m \in \mathbb{N}$.*

Intermediate series $\mathcal{L}$-modules are intermediate series Vir-modules with trivial action of W_n. For $\alpha, \beta \in \mathbb{C}$, define $V_{\alpha,\beta,0} = \operatorname{span}_{\mathbb{C}}\{v_m : m \in \mathbb{Z}\}$ with

$$L_n v_m = -(m + \alpha + \beta + n\beta)v_{m+n}, \quad C v_m = W_n v_m = 0,$$

for $n, m \in \mathbb{Z}$. Then $V_{\alpha,\beta,0} \cong V_{\alpha+k,\beta,0}$ for $k \in \mathbb{Z}$, so when $\alpha \in \mathbb{Z}$ we may assume $\alpha = 0$. Module $V_{\alpha,\beta,0}$ is reducible if and only if $\alpha \in \mathbb{Z}$ and $\beta \in \{0, 1\}$. Define $V'_{0,0,0} := V_{0,0,0}/\mathbb{C}v_0$, $V'_{0,1,0} := \bigoplus_{m\neq -1} \mathbb{C}v_m \subseteq V_{0,1,0}$ and $V'_{\alpha,\beta,0} = V_{\alpha,\beta,0}$ otherwise. Then $V'_{\alpha,\beta,0}$ are all irreducible modules from intermediate series. [LZ]

Theorem 2.2 ([LZ]) *An irreducible weight $\mathcal{L}$-module with finite-dimensional weight spaces is either highest (or lowest) weight module, or isomorphic to $V'_{\alpha,\beta,0}$ for some $\alpha, \beta \in \mathbb{C}$.*

Tensor product $V'_{\alpha,\beta,0} \otimes L(c,h,h_W)$ carries an $\mathcal{L}$-module structure with action

$$x(v_n \otimes v) := xv_n \otimes v + v_n \otimes xv, \text{ for any } x \in \mathcal{L}.$$

Note that $W_m(v_n \otimes v) = v_n \otimes W_m v$. Like in Vir case, $\{v_n \otimes v : n \in \mathbb{Z}\}$ generates $V'_{\alpha,\beta,0} \otimes L(c,h,h_W)$ (see (4) in Lemma 5.5). Moreover, all weight subspaces are infinite-dimensional:

$$\left(V'_{\alpha,\beta} \otimes L\left(c,h\right)\right)_{m-\alpha-\beta} = \bigoplus_{n \in \mathbb{Z}_+} \mathbb{C}v_{n-m} \otimes L\left(c,h\right)_n$$

3 Verma module structure

In this section we assume $2h_W + \frac{p^2-1}{12}c = 0$ for some $p \in \mathbb{N}$.

Some results of this section are similar to those presented in [JP], and motivated by [B]. However, since W_0 does not act semisimply in general (unlike to I_0 in Heisenberg-Virasoro algebra), submodules generated by some singular vectors in Verma modules are not isomorphic to Verma modules, so the maximal submodule $J(c,h,h_W)$ is not necessarily cyclic on a singular vector. In fact, we prove existence of subsingular vectors in some Verma modules. Therefore, Corollary 3.6 and later results in [JP] are not correct in general.

Using determinant formula, it was proved in [JP] (Proposition 3.1), that if $2h_W + \frac{p^2-1}{12}c = 0$ for some $p \in \mathbb{N}$, then a singular vector $u \in V(c,h,h_W)_p$ exists.

We state the main result of this section, a structure theorem for Verma modules over the Lie algebra $W(2,2)$.

Theorem 3.1 *Let $2h_W + \frac{p^2-1}{12}c = 0$ for some $p \in \mathbb{N}$. Then a singular vector $u' \in V(c,h,h_W)_{h+p} \cap \mathcal{W}$ such that $\overline{u'} = W_{-p}v$ exists, and $U(\mathcal{L})u' \cong V(c,h+p,h_W)$. Images of vectors $W_{-m_s} \cdots W_{-m_1} L_{-n_t} \cdots L_{-n_1} v$, such that $m_i \neq p$ form a PBW basis for $L'(c,h,h_W) := V(c,h,h_W)/U(\mathcal{L})u'$. Moreover:*

(i) *Let $h \neq h_W + \frac{(13p+1)(p-1)}{12} + \frac{(1-r)p}{2}$ for all $r \in \mathbb{N}$. Then $U(\mathcal{L})u' = J(c,h,h_W)$ is a maximal submodule in $V(c,h,h_W)$, and $L'(c,h,h_W) = L(c,h,h_W)$ is irreducible.*

(ii) *Suppose $L'(c,h,h_W)$ is reducible. Then a subsingular vector $u \in V(c,h,h_W)$ such that $\overline{u} = L^r_{-p}$ for some $r \in \mathbb{N}$ exists. Vectors u and u' generate maximal submodule $J(c,h,h_W)$, and images of $W_{-m_s} \cdots W_{-m_1} L_{-n_t} \cdots L_{-n_1} v$ in which neither W_{-p} nor L^k_{-p} for $k \geq r$ occur as factors, form a PBW basis for irreducible module $L(c,h,h_W) = V(c,h,h_W)/J(c,h,h_W)$.*

We suspect $L'(c,h,h_W)$ is reducible if and only if $h = h_W + \frac{(13p+1)(p-1)}{12} + \frac{(1-r)p}{2}$. Proof of Theorem 3.1 will be presented in a series of lemmas and theorems in the rest of this section. We follow the Billig's idea from [B], also used in [JP].

Jiang and Pei introduced W-degree on $\mathcal{L}_-$:

$$\deg_W L_{-n} = 0, \quad \deg_W W_{-n} = 1,$$

which induces $\mathbb{Z}$-grading on $U(\mathcal{L})$ and on $V(c, h, h_W)$

$$\deg_W W_{-m_s} \cdots W_{-m_1} L_{-n_t} \cdots L_{-n_1} v = s.$$

Obviously, this grading depends on a basis.

For a nonzero $x \in V(c, h, h_W)$ we denote by $\overline{x}$ its lowest nonzero homogeneous component with respect to W-degree (in a standard PBW basis). If $x \in V^W_{(k)}(c, h, h_W)$ and $n > 0$, then

$$\begin{aligned} W_n x &\in V(c, h, h_W)^W_{(k)} \oplus V(c, h, h_W)^W_{(k+1)} \\ L_n x &\in V(c, h, h_W)^W_{(k-1)} \oplus V(c, h, h_W)^W_{(k)}. \end{aligned} \tag{2}$$

We can define L-degree, likewise:

$$\deg_W L_{-n} = 1, \quad \deg_W W_{-n} = 0.$$

Let

$$\mathcal{W} = V^L_{(0)}(c, h, h_W) = \{W_{-m_s} \cdots W_{-m_1} v : m_s \geq \cdots \geq m_1 > 0\}.$$

The following lemma from [JP] will be often used throughout this section (see also analogous result in [B]).

Lemma 3.2 ([JP]) *Let $0 \neq x \in V(c, h, h_W)$ and $\deg_W \overline{x} = k$.*

a) *If $\overline{x} \notin \mathcal{W}$ and $n \in \mathbb{N}$ is the smallest, such that L_{-n} occurs as a factor in one of the terms in $\overline{x}$, then the part of $W_n x$ of the W-degree k is given by*

$$n\left(2h_W + \frac{n^2 - 1}{12} c\right) \frac{\partial \overline{x}}{\partial L_{-n}}.$$

b) *If $\overline{x} \in \mathcal{W}$, $\overline{x} \notin \mathbb{C}v$ and $m \in \mathbb{N}$ is maximal, such that W_{-m} occurs as a factor in one of the terms of $\overline{x}$, then the part of $L_m x$ of the W-degree $k - 1$ is given by*

$$m\left(2h_W + \frac{m^2 - 1}{12} c\right) \frac{\partial \overline{x}}{\partial W_{-m}}.$$

As a first application of this lemma, we have the following result from [JP].

Lemma 3.3 ([JP]) *Let $2h_W + \frac{p^2-1}{12} c = 0$. Then there is a singular vector $u \in V(c, h, h_W)_{h+p}$ such that $\overline{u} = W_{-p} v$ or $\overline{u} = L_{-p} v$.*

Now we define u_n inductively. Let $u_0 = u$, and $u_n = u_{n-1} - \overline{u_{n-1}}$. Then

$$u = \sum_{n\geq 0} \overline{u_n}, \text{where } \overline{u_0} = L_{-p}v \text{ ili } W_{-p}v.$$

The sum above is a decomposition of u by W-degree, i.e. $\deg_W \overline{u_i} < \deg_W \overline{u_j}$ if $i < j$. Moreover, $\overline{u_n}$ is homogeneous respective to W-degree (all components of $\overline{u_n}$ have the same W-degree). For example $u = \underbrace{L_{-3}v}_{\overline{u_0}} + \overbrace{\underbrace{W_{-3}v + W_{-1}L_{-2}v}_{\overline{u_1}} + \overbrace{\underbrace{W_{-1}^3 v}_{\overline{u_2}}}^{u_2}}^{u_1}$.

Theorem 3.4 *Let $2h_W + \frac{p^2-1}{12}c = 0$. Then there is a singular vector $u' \in V(c,h,h_W)_p \cap \mathcal{W}$, such that $\overline{u} = W_{-p}v$. Moreover, $U(\mathcal{L})u'$ is isomorphic to Verma module $V(c,h+p,h_W)$.*

Proof. We know from Lemma 3.3, that there exists a singular vector in $V(c,h,h_W)_p$. If $\overline{u} = W_{-p}v$ we set $u' = u$. Suppose that $\overline{u} = L_{-p}v$. then $W_0 u = h_W u + pu'$ and u' is obviously a singular vector such that $\overline{u'} = W_{-p}v$.

Now we show that $u' \in \mathcal{W}$. Let $n \in \mathbb{N}$ the smallest such that, for some i, L_{-i} occurs as a factor in $\overline{u'_n}$, and let $\deg_W \overline{u'_n} = k$. Moreover, let $m \in \mathbb{N}$ the smallest, such that L_{-m} occurs as a factor in $\overline{u'_n}$. Obviously, $m < p$. Then by Lemma 3.2 a) the part of $W_m u'_n$ of the W-degree k is nonzero. However, $W_m u'_n = W_m u' - W_m \overline{u'_{n-1}} = 0$ because u' is a singular vector, and $\overline{u_{n-1}} \in \mathcal{W}$ by minimality of n, so we got a contradiction and conclude $\deg_L u' = 0$.

Finally, since $u \in \mathcal{W}$, we have $W_0 u' = h_W u'$ so $U(\mathcal{L})\,u$ is a Verma module with the highest weight $(c, h+p, h_W)$. ∎

Example 3.5 *$u' = W_{-1}v$ is a singular vector in the Verma module $V(c,h,0)$;*

$u' = (W_{-2} - \frac{3}{4h_W}W_{-1}^2)v$ is a singular vector in the Verma module $V(c,h,-c/8)$;

$u' = (W_{-3} - \frac{2}{h_W}W_{-2}W_{-1} + \frac{1}{h_W^2}W_{-1}^3)v$ is a singular vector in the Verma module $V(c,h,-c/3)$.

From now on, we use the following notation

$$J'(c,h,h_W) := U(\mathcal{L})\,u,$$
$$L'(c,h,h_W) = V(c,h,h_W)/J'(c,h,h_W).$$

Let $P_2(n) = \sum_{i=0}^{n} P(n-i)P(i)$, where P is Kostant partition function, with $P(0) = 1$. Then

$$\operatorname{char} V(c,h,h_W) = q^h \sum_{n\geq 0} P_2(n)q^n = q^h \prod_{k\geq 1}(1-q^k)^{-2}.$$

From Theorem 3.4 we get

$$\begin{aligned}
\operatorname{char} J'(c,h,h_W) &= q^{h+p}\sum_{n\geq 0} P_2(n)q^n = q^{h+p}\prod_{k\geq 1}(1-q^k)^{-2},\\
\operatorname{char} L'(c,h,h_W) &= \operatorname{char} V - \operatorname{char} J = q^h(1-q^p)\sum_{n\geq 0} P_2(n)q^n = \\
&= q^h(1-q^p)\prod_{k\geq 1}(1-q^k)^{-2}.
\end{aligned}$$

Lemma 3.6 *Let $0\neq x\in J'(c,h,h_W)$. Then there exist terms in $\overline{x}$, containing factor W_{-p}.*

Proof. We may write $x = yu'$, where $y\in U(\mathcal{L}_-)$. Since $U(\mathcal{L}_-)$ has no zero divisors, we get $\overline{x} = \overline{y}\overline{u'}$. However, $\overline{u'} = W_{-p}v$ so every component in $\overline{x}$ with maximal length contains W_{-p}. ■

Lemma 3.7 *Denote by B' set of PBW vectors $W_{-m_s}\cdots W_{-m_1}L_{-n_t}\cdots L_{-n_1}v$ modulo $J'(c,h,h_W)$, with $m_i\neq p$. Then B' is a basis for $L'(c,h,h_W)$.*

Proof. Since W_{-p} cannot occur in a linear combination of vectors from B', it follows from Lemma 3.6 that B' is linearly independent. Simple combinatorics shows that a character of a vector space spanned by B' equals $q^h(1-q^p)\sum_{n\geq 0}P_2(n)q^n$, which proves B' is basis for $L'(c,h,h_W)$. ■

Remark 3.8 *If a singular vector $u\in V(c,h,h_W)$ such that $\overline{u} = L_{-p}v$ exists, then $L'(c,h,h_W)$ is reducible, and the image of u in $L'(c,h,h_W)$ is a singular vector. Next we show that $L'(c,h,h_W)$ may have singular vectors of weight $h+pr$ for any $r\in\mathbb{N}$, i.e. $V(c,h,h_W)$ may have a subsingular vector. Singular vector u is just a special case $r=1$.*

Theorem 3.9 *If $L'(c,h,h_W)$ is reducible, then there is $u\in L'(c,h,h_W)$ such that $\overline{u} = L_{-p}^r v$ for some $r\in\mathbb{N}$.*

Proof. Let $u\in L'(c,h,h_W)\setminus\mathbb{C}v$ homogeneous vector such that $\mathcal{L}_+u=0$ (i.e. $\mathcal{L}_+u\in J'(c,h,h_W)$ if we consider u as a vector in $V(c,h,h_W)$). We may assume u is a linear combination of vectors from B'. Note that B' is closed under partial derivations $\frac{\partial}{\partial L_{-n}}$ and $\frac{\partial}{\partial W_{-n}}$.

Suppose $\overline{u}\in\mathcal{W}$ and $\overline{u}\notin\mathbb{C}v$. Applying Lemma 3.2 b) we find $m\in\mathbb{N}$, $m\neq p$, such that $\overline{L_m u}\neq 0$ and W_{-p} does not occur in $\overline{L_m u}$. By Lemma 3.6, $L_m u\notin J'(c,h,h_W)$, a contradiction.

Therefore we may assume $\overline{u}\notin\mathcal{W}$. Let $\deg_W \overline{u} = k$ and let n the smallest such that L_{-n} occurs in $\overline{u}$. By Lemma 3.2 a), component of $W_n u$ of the W-degree k is $n\left(2h_W + \frac{n^2-1}{12}c\right)\frac{\partial\overline{u}}{\partial L_{-n}}$. If $n\neq p$, we get contradiction again, so $n=p$. Let $n'>p$ the smallest such that $L_{-n'}$ occurs as a factor in $\overline{u}$. Since

$$W_{n'}u = \sum_{i=1}^{t} W_{-m_k}\cdots W_{-m_1}L_{-n_t}\cdots[W_{n'},L_{-n_i}]\cdots L_{-n_1}v$$

and $n_1 = p$ or $n_1 \geq n'$ the only component of $\overline{u}$ that produces part of W-degree k is one in which $L_{-n'}$ occurs. That component equals $n'\left(2h_W + \frac{n'^2-1}{12}c\right)\frac{\partial \overline{u}}{\partial L_{-n'}}$. Since $n' \neq p$ and since W_{-p} does not occur as a factor in $\overline{u}$, we get $0 \neq W_{n'}u \in J'(c,h,h_W)$ such that W_{-p} does not occur in $\overline{W_{n'}u}$, again a contradiction with Lemma 3.6. Therefore if L_{-t} occurs as a factor in $\overline{u}$, then $t = p$.

It is left to prove that $\overline{u} \in \mathbb{C}L_{-p}^r v$, i.e. that $\deg_W \overline{u} = 0$. Suppose that $\deg_W \overline{u} = k$, and let m the highest such that W_{-m} occurs as a factor in $\overline{u}$. It is easy to see that part of $L_m u$ of W-degree $k-1$ equals $m\left(2h_W + \frac{m^2-1}{12}c\right)\frac{\partial \overline{u}}{\partial W_{-m}}$. Since, by Lemma 3.6, $m \neq p$, we conclude that $0 \neq L_m u \in J'(c,h,h_W)$ such that W_{-p} does not occur as a factor in $\overline{L_m u}$, a contradiction. this proves that $\overline{u}$ equals $L_{-p}^r v$ up to a scalar factor, so we set $\overline{u} = L_{-p}^r v$. ■

Let us show that subsingular vector can exist.

Example 3.10 *$u = L_{-1}v$ is a subsingular vector in the Verma module $V(c,0,0)$; $u = (L_{-1}^2 + \frac{6}{c}W_{-2})v$ is a subsingular vector in the Verma module $V(c,-\frac{1}{2},0)$.*

Our next goal is to find a necessary condition for existence of a subsingular vector. We still assume that $2h_W + \frac{p^2-1}{12}c = 0$. Suppose a subsingular vector u exists in $V(c,h,h_W)$, such that $\overline{u} = L_{-p}^r v$. Consider $L_p u \in J'(c,h,h_W)$. Obviously, a component

$$L_p L_{-p}^r v = \left(n\frac{p^3-p}{12}c + 2p\sum_{i=1}^{r-1}(h+ip)\right)L_{-p}^{r-1}v = rp(2h - 2h_W + (r-1)p)L_{-p}^{r-1}v$$

occurs in $L_p u$. Since $L_p u \in J'(c,h,h_W)$, by Lemma 3.6, a coefficient with $L_{-p}^{r-1}v$ in $L_p u$ has to be zero. The only components of u that contribute to $L_{-p}^{r-1}v$ in $L_p u$ are $\lambda_i W_{-i} L_{-p}^{r-1} L_{-(p-i)} v$ for $i = 1, \ldots, p-1$, and their contribution is

$$\lambda_i (p+i)(p-i)\left(2h_W + \frac{(p-i)^2-1}{12}c\right)L_{-p}^{r-1}v = 2\lambda_i h_W i(2p-i)\frac{p^2-i^2}{p^2-1}L_{-p}^{r-1}v.$$

It is left to find λ_i. Consider $L_i u$ for $i = 1, \ldots, p-1$. Vector $\lambda_i L_i W_{-i} L_{-p}^{r-1} L_{-(p-i)} v$ produces a component

$$2\lambda_i h_W i\frac{p^2-i^2}{p^2-1}L_{-p}^{r-1}L_{-(p-i)}v.$$

The only other component contributing to $L_{-p}^{r-1}L_{-(p-i)}v$ in $L_i u$ is $L_{-p}^r v$, with a coefficient $n(p+i)L_{-p}^{r-1}L_{-(p-i)}v$. Since $L_{-p}^{r-1}L_{-(p-i)}v$ can not occur in a vector from $J'(c,h,h_W)$, associated coefficient needs to be zero, which gives $\lambda_i = -r\frac{p^2-1}{2h_w i(p-i)}$. Now we have a formula for a coefficient of L_{-p}^{r-1}:

$$rp(2h - 2h_W + (r-1)p) - 2rh_W\sum_{i=1}^{p-1} i(2p-i)\frac{p^2-i^2}{p^2-1}\frac{p^2-1}{2h_w i(p-i)} = 0$$

leading to

$$h = h_W + \frac{(13p+1)(p-1)}{12} + \frac{(1-r)p}{2}.$$

This proves the following

Theorem 3.11 *Let $2h_W + \frac{p^2-1}{12}c = 0$. If $V(c,h,h_W)$ contains a subsingular vector u such that $\overline{u} = L_{-p}^r v$, for some $r \in \mathbb{N}$, then $h = h_W + \frac{(13p+1)(p-1)}{12} + \frac{(1-r)p}{2}$.*

From Theorems 3.9 and 3.11, and Lemma 3.7, we get:

Corollary 3.12 *Suppose $2h_W + \frac{p^2-1}{12}c = 0$. If $h \neq h_W + \frac{(13p+1)(p-1)}{12} + \frac{(1-r)p}{2}$ for all $r \in \mathbb{N}$, then $J'(c,h,h_W) = J(c,h,h_W)$ is a maximal submodule in $V(c,h,h_W)$, and $L'(c,h,h_W) = L(c,h,h_W)$ is irreducible module with PBW basis*

$$B' = \{W_{-m_s} \cdots W_{-m_1} L_{-n_t} \cdots L_{-n_1} v : m_j \neq p\}$$

and a character

$$\operatorname{char} L'(c,h,h_W) = q^h(1-q^p) \sum_{n \geq 0} P_2(n) q^n = q^h (1-q^p) \prod_{k \geq 1} (1-q^k)^{-2}.$$

Assume now that a subsingular vector $u \in V(c,h,h_W)$ exists, such that $\overline{u} = L_{-p}^r v$. We show $U(\mathcal{L})\{u, u'\} = J(c,h,h_W)$ is a maximal submodule in $V(c,h,h_W)$.

First we need a generalization of Lemma 3.2 to $L'(c,h,h_W)$.

Lemma 3.13 *Let $x = W_{-m_k} \cdots W_{-m_1} L_{-n_t} \cdots L_{-n_1} v \in L'(c,h,h_W)$ such that $m_j = p$ for some j, and $m_i \neq p$ for $i \neq j$. Then $\deg_W \overline{x} \geq k$, if x is considered in a basis B'.*

Proof. We make use of relation $u' = 0$ in $L'(c,h,h_W)$. This leads to $W_{-p}v = \sum_i z_i v$, where $\deg_W z_i > 1$. Then $W_{-p} L_{-n} v = (n-p) W_{-p-n} v + \sum z'_i v$, where $\deg_W z'_i = \deg_W z_i > 1$. We continue by induction to prove $W_{-p} L_{-n_t} \cdots L_{-n_1} v = \sum y_i v$, where $y_i v \in B'$ and $\deg_W y_i > 1$. Acting with W_{-m_i} for $i \neq j$ we get $W_{-m_k} \cdots W_{-m_1} L_{-n_t} \cdots L_{-n_1} v = \sum W_{-m_k} \cdots W_{-m_1} y_i v$. ■

Lemma 3.14 *Let $0 \neq x \in L'(c,h,h_W)$ and $\deg_W \overline{x} = k$. If $\overline{x} \notin \mathcal{W}$ and $n \in \mathbb{N}$ is the smallest, such that L_{-n} occurs as a factor in one of the terms in $\overline{x}$, then the part of $W_n x$ of the W-degree k is given by*

$$n \left(2h_W + \frac{n^2-1}{12} c \right) \frac{\partial \overline{x}}{\partial L_{-n}}.$$

Proof. We see from (2) that the part of W-degree k comes from $W_n\overline{x}$. Let $y = W_{-m_k} \cdots W_{-m_1} L_{-n_t} \cdots L_{-n_1} v$, $m_j \neq p$, a component of $\overline{x}$. Then

$$W_n y = \sum_{i=1}^{t} W_{-m_k} \cdots W_{-m_1} L_{-n_t} \cdots [W_n, L_{-n_i}] \cdots L_{-n_1} v$$

and $n_t \geq n$. If $n_i = n$, we have

$$[W_n, L_{-n}] = n \left(2W_0 + \frac{n^2-1}{12} C \right)$$

so this part contributes with $n(2h_W + \frac{n^2-1}{12}c)\frac{\partial \overline{x}}{\partial L_{-n}}$.

If $n_i > n$, we get

$$[W_n, L_{-n_i}] = (n+n_i) W_{n-n_i}.$$

In case $n - n_i \neq p$ we get a component with W-degree $k+1$. Suppose $n - n_i = p$. Then by Lemma 3.13, this component can be substituted by sum of components with W-degree $> k$. This completes the proof. ■

Lemma 3.15 *If u is a subsingular vector such that $\overline{u} = L^r_{-p} v$, then* $\deg_L u = r$.

Proof. Consider u as a singular vector in $L'(c, h, h_W)$. We use notation $u = \sum_{n \geq 0} \overline{u_n}$, where $\overline{u_0} = L^r_{-p}$, and proceed by induction on n. Suppose $\deg_L \overline{u_j} \leq r$ for $j < n$. Suppose to the contrary, that $\deg_L \overline{u_n} > r$. Denote by x the sum of all components in u_n of L-degree at most r and $u_n = x + y$. Note that by definition, $\overline{u_n}$ is homogeneous respective to W-degree. Say $\deg_W \overline{u_n} = k$. Then $\deg_W \overline{y} = k$. Let m the smallest such that L_{-m} occurs as a factor in y. Obviously, $m < p$. We have $y = u - \sum_{i=0}^{n-1} \overline{u_i} - x$. By Lemma 3.14 the part of $W_m y$ of the W-degree k is $m(2h_W + \frac{m^2-1}{12}c)\frac{\partial \overline{y}}{\partial L_{-m}} \neq 0$. Since $\deg_L y > r$, we have $\deg_L W_m y \geq r$. On the other hand $\deg_L(\sum_{i=0}^{n-1} \overline{u_i} - x) \leq r$ so $\deg_L W_m(\sum_{i=0}^{n-1} \overline{u_i} - x) < r$. Since $W_m u = 0$ in $L'(c, h, h_W)$, we have $W_m y = -W_m(\sum_{i=0}^{n-1} \overline{u_i} - x)$, which is a contradiction. ■

Lemma 3.16 *Let $0 \neq x \in U(\mathcal{L})\{u, u'\}$. Then W_{-p} or L^k_{-p} for some $k \geq r$ occur as a factor in some part of $\overline{x}$.*

Proof. Suppose x is homogeneous. Let $x = yu + zu'$. If $y \in U(\mathcal{L}_+)$, $x \in U(\mathcal{L})u'$, so we may apply Lemma 3.6.

Furthermore, vectors from $\mathcal{L}_0 u$ contain $L^r_{-p} v$ since

$$W_0 L^r_{-p} v = h_W L^r_{-p} + rpW_{-p} L^{r-1}_{-p} v,$$

so $\overline{x} \in \mathbb{C} L^r_{-p}$. Finally, for $y \in U(\mathcal{L}_-)$, x must contain L^k_{-p} for some $k \geq r$ as a factor in the longest component of $\overline{x}$. ■

For a PBW monomial $x = W_{-m_s} \cdots W_{-m_1} L_{-n_t} \cdots L_{-n_1} v$ define L_{-p}-degree as a number of factors $L_{-n_i} = L_{-p}$, i.e. $\deg_{L_{-p}} x = |\{i \in \{1, \ldots, t\} : n_i = p\}|$.

Lemma 3.17 *Let B the set of all PBW vectors $W_{-m_s}\cdots W_{-m_1}L_{-n_t}\cdots L_{-n_1}v$, such that $m_j\neq p$, and with L_{-p}-degree less than r. Then B is a basis for $V(c,h,h_W)/U(\mathcal{L})\{u,u'\}$.*

Proof. From Lemma 3.16 follows linear independence of B. Next we show that $a\in V(c,h,h_W)/U(\mathcal{L})\{u,u'\}$ can be represented in B. Since $U(\mathcal{L})\{u,u'\}$ contains $J'(c,h,h_W)$, we can represent a in B', i.e. without W_{-p} as a factor. Suppose a contains monomial $W_{-m_s}\cdots W_{-m_1}L_{-n_t}\cdots L_{-n_1}v$ with L_{-p}-degree greater than $r-1$. We use relation

$$0=u=L_{-p}^r v+\sum x_i,\quad \deg_L x_i\leq r, \deg_{L_{-p}} x_i<r \tag{3}$$

to eliminate every occurrence of a factor L_{-p}^r.

Step 1 Acting with $W_{-m_s}\cdots W_{-m_1}$ on (3) we immediately get

$$W_{-m_s}\cdots W_{-m_1}L_{-p}^r v=\sum W_{-m_s}\cdots W_{-m_1}x_i$$

and L_{-p}^r can not occur on the right side.

Step 2 Let $n_1>p$. Acting with $L_{-n_t}\cdots L_{-n_1}$ on (3) we get

$$L_{-n_t}\cdots L_{-n_1}L_{-p}^r v=\sum y_i,\quad \deg_{L_{-p}} y_i=\deg_{L_{-p}} x_i$$

because $[L_{-n},L_{-m}]$ can not produce L_{-p} if $n>p$. So L_{-p}^r does not occur on the right side.

Step 3 Let $n\leq p$. Acting with L_{-n} on (3) we get

$$L_{-p}^r L_{-n}v=\sum z_i,\quad \deg_L z_i\leq r+1, \deg_{L_{-p}} z_i\leq r.$$

the only part that can produce L_{-p}^r comes from $L_{-n}wL_{-p}^{r-1}L_{-(p-n)}v$ for some $w\in\mathcal{W}$ (because $\deg_L u=r$). From this we get $(p-2n)wL_{-p}^r v$ and we may apply Step 1 to eliminate this component.

Proceed by induction. Suppose that for every $j_1,\ldots,j_{n-1}$, there exist u_i, such that $\deg_L u_i\leq r+n-1$, $\deg_{L_{-p}} u_i<r$ and $L_{-p}^r L_{-j_{n-1}}\cdots L_{-j_1}v=\sum u_i$. Acting with L_{-j_n} we get

$$L_{-p}^r L_{-j_n}\cdots L_{-j_1}v=\sum w_i,\quad \deg_L w_i\leq r+n, \deg_{L_{-p}} w_i\leq r.$$

Part on the right side that can produce L_{-p}^r has to come from monomial $L_{-n}wL_{-p}^{r-1}L_{-k_l}\cdots L_{-k_1}v$ with $l\leq n$ for some $w\in U(\mathcal{L}_-)$. From this we get $w'L_{-p}^r L_{-k'_{l-1}}\cdots L_{-k'_1}v$ and by induction this component can be replaced with a vector from B.

This completes the proof. ■

Theorem 3.18 *Let $2h_W + \frac{p^2-1}{12}c = 0$. If $V(c,h,h_W)$ contains a subsingular vector u such that $\overline{u} = L_{-p}^r v$, for some $r \in \mathbb{N}$, then $J(c,h,h_W) = U(\mathcal{L})\{u,u'\}$ is the maximal submodule, and $L(c,h,h_W) = V(c,h,h_W)/J(c,h,h_W)$ is irreducible module with PBW basis*

$$B = \left\{ x = W_{-m_s} \cdots W_{-m_1} L_{-n_t} \cdots L_{-n_1} v : m_j \neq p, \deg_{L_{-p}} x < r \right\}$$

and a character

$$\operatorname{char} L(c,h,h_W) =$$

$$= q^h (1-q^p)(1-q^{rp}) \sum_{n \geq 0} P_2(n) q^n = q^h (1-q^p)(1-q^{rp}) \prod_{k \geq 1} (1-q^k)^{-2}.$$

Proof. Let $0 \neq x \in V(c,h,h_W)$ such that $U(\mathcal{L}_+)x \subseteq J(c,h,h_W)$. We may view x as a homogeneous vector presented in B. Analogous to proof of Theorem 3.9, one can show that $\overline{x} = \mathbb{C} L_{-p}^s v$ for some $s \in \mathbb{N}$. From Lemma 3.17 we conclude $s < r$ so we have found $x \in V(c,h,h_W)$, such that $U(\mathcal{L}_+)x \in U(\mathcal{L})u'$ and $\overline{x} = L_{-p}^s v$ for $s < r$. This is a contradiction to Theorem 3.11, proving irreducibility of $L(v,h,h_W)$.

Simple combinatorics on B shows that

$$\operatorname{char} L(c,h,h_W) =$$

$$= q^h \left(\sum_{n \geq 0} P_2(n) q^n - q^p \sum_{n \geq 0} P_2(n) q^n - q^{rp} \sum_{n \geq 0} P_2(n) q^n + q^{(r+1)p} \sum_{n \geq 0} P_2(n) q^n \right)$$

$$= q^h (1-q^p)(1-q^{rp}) \sum_{n \geq 0} P_2(n) q^n = q^h (1-q^p)(1-q^{rp}) \prod_{k \geq 1} (1-q^k)^{-2}$$

so all claims are proved. ■

This completes the proof of Theorem 3.1.

Note that $\operatorname{char} J(c,h,h_W) = \operatorname{char} V(c,h,h_W) - \operatorname{char} L(c,h,h_W)$ so

$$\begin{aligned} \operatorname{char} J(c,h,h_W) &= q^{h+p}(1 + q^{(r-1)p} - q^{rp}) \sum_{n \geq 0} P_2(n) q^n = \\ &= q^{h+p}(1 + q^{(r-1)p} - q^{rp}) \prod_{k \geq 1} (1-q^k)^{-2}. \end{aligned}$$

From there we get

$$\begin{aligned} \operatorname{char} J(c,h,h_W)/J'(c,h,h_W) &= q^{h+rp}(1-q^p) \sum_{n \geq 0} P_2(n) q^n = \\ &= q^{h+rp}(1-q^p) \prod_{k \geq 1} (1-q^k)^{-2} \end{aligned}$$

so

$$J(c,h,h_W)/J'(c,h,h_W) \cong L'(c,h+rp,h_W) = L(c,h+rp,h_W).$$

Last equation is due to

$$h + rp = h_W + \frac{(13p+1)(p-1)}{12} + \frac{(1+r)p}{2}, r \in \mathbb{N}$$

and Corollary 3.12.

Since $u \bmod J'(c,h,h_W)$ generates $J(c,h,h_W)/J'(c,h,h_W)$, we get $W_0 u \in h_W u + J'(c,h,h_W)$.

Likewise,

$$\operatorname{char} L'(c,h,h_W)/L(c,h+rp,h_W) = \operatorname{char} L(c,h,h_W).$$

Corollary 3.19 *If Verma module $V(c,h,h_W)$ contains a subsingular vector u, such that $\overline{u} = L_{-p}^r v$, then the following short sequence is exact*

$$0 \to L(c,h+rp,h_W) \to L'(c,h,h_W) \to L(c,h,h_W) \to 0.$$

Since three modules are irreducible, $L(c,h+rp,h_W)$ is the only nontrivial submodule in $L'(c,h,h_W)$.

Next we present examples of singular and subsingular vectors in simplest cases.

Singular vector $u' \in \mathcal{W}$:

$h_W = 0$	$W_{-1}v$
$c = -8h_W$	$\left(W_{-2} - \frac{3}{4h_W}W_{-1}^2\right)v$
$c = -3h_W$	$\left(W_{-3} - \frac{2}{h_W}W_{-2}W_{-1} + \frac{1}{h_W^2}W_{-1}^3\right)v$
$c = -\frac{8}{5}h_W$	$\left(W_{-4} - \frac{5}{2h_W}W_{-3}W_{-1} - \frac{15}{16h_W}W_{-2}^2 + \right.$ $\left. + \frac{125}{32h_W^2}W_{-2}W_{-1}^2 - \frac{375}{256h_W^3}W_{-1}^4\right)v$
$c = -h_W$	$(W_{-5} - \frac{3}{h_W}W_{-4}W_{-1} - \frac{2}{h_W}W_{-3}W_{-2} + \frac{21}{4h_W^2}W_{-3}W_{-1}^2 +$ $+ \frac{3}{h_W^2}W_{-2}^2W_{-1} + \frac{13}{2h_W^3}W_{-2}W_{-1}^3 - \frac{39}{20h_W^3}W_{-1}^5)v$

Let $r = 1$. Then subsingular vector u is actually singular vector on level p. By Lemma 3.15 we can write $u = w_0 v + \sum_{i=1}^{p-1} w_i L_{-i} v + L_{-p} v$, where $w_i \in U\left(\mathcal{L}_-\right)_{p-i}$, $\deg_L w_i = 0$. Acting with W_{p-n} we get recursive relation for w_n, $n = 1, \ldots, p-1$. We use identity $[W_n, L_{-n}]v = 2h_W n \frac{n^2-p^2}{1-p^2} v$ since $2h_W + \frac{p^2-1}{12}c = 0$.

$$\begin{aligned} 0 = W_{p-1}u &= w_{p-1}[W_{p-1}, L_{-(p-1)}]v + [W_{p-1}, L_{-p}]v = \\ &= \frac{2p-1}{p+1} 2h_W w_{p-1} v + (2p-1)\, W_{-1} v \end{aligned}$$

so $w_{p-1} = -\frac{p+1}{2h_W}W_{-1}$. Next we act with W_{p-2} to get w_{p-2} and so on. In general:

$$w_n = \frac{p^2-1}{2h_W n(n^2-p^2)} \left(\sum_{i=n+1}^{p-1} (n+i)\, w_i W_{-(i-n)} + (n+p)\, W_{-(p-n)} \right).$$

Since $W_0 u = h_W u + p u'$ (we put scalar p to have $\overline{u'} = W_{-p} v$), we may freely chose coefficient of $W_{-p} v$ in w_0, so we assume $W_{-p} v$ does not occur in u.

(Sub)singular vector u, when $r = 1$:

$h = h_W = 0$	$L_{-1} v$
$h = h_W + \frac{9}{4}$	$\left(L_{-2} - \frac{3}{2h_W} W_{-1} L_{-1} + \frac{12h_W+39}{16h_W^2} W_{-1}^2\right) v$
$h = h_W + \frac{20}{3}$	$\Big(L_{-3} - \frac{2}{h_W} W_{-1} L_{-2} - \frac{2}{h_W} W_{-2} L_{-1} + \frac{3}{h_W^2} W_{-1}^2 L_{-1}$ $+ \frac{58}{3h_W^2} W_{-2} W_{-1} - \frac{2h_w+52}{3h_W^3} W_{-1}^3\Big) v$
$h = h_W + \frac{53}{4}$	$\Big(L_{-4} - \frac{5}{2h_W} W_{-1} L_{-3} - \frac{15}{8h_W} W_{-2} L_{-2} + \frac{125}{32h_W^2} W_{-1}^2 L_{-2}$ $- \frac{5}{2h_W} W_{-3} L_{-1} + \frac{125}{16h_W^2} W_{-2} W_{-1} L_{-1} - \frac{375}{64h_W^3} W_{-1}^3 L_{-1}$ $+ \frac{325+20h_W}{8h_W^2} W_{-3} W_{-1} + \frac{8125}{64h_W^3} W_{-2} W_{-1}^2$ $+ \frac{975+60h_W}{64h_W^2} W_{-2}^2 + \frac{1125}{256h_W^3}\left(\frac{65}{4h_W} + 1\right) W_{-1}^4\Big) v$

Using computer, we have found formulas for singular vectors u up to level 8.

Subsingular vectors u in $V(c, \frac{1-r}{2}, 0)$:

$h = h_W = 0$	$L_{-1} v$
$h = -\frac{1}{2}$	$\left(L_{-1}^2 + \frac{6}{c} W_{-2}\right) v$
$h = -1$	$\left(L_{-1}^3 + \frac{12}{c} W_{-3} + \frac{24}{c} W_{-2} L_{-1}\right) v$
$h = -\frac{3}{2}$	$\left(L_{-1}^4 + \frac{36}{c} W_{-4} + \frac{60}{c} W_{-3} L_{-1} + \frac{108}{c^2} W_{-2}^2 + \frac{60}{c} W_{-2} L_{-1}^2\right) v$
$h = -2$	$\Big(L_{-1}^5 + \frac{144}{c} W_{-5} + \frac{48}{c} W_{-4} L_{-1} + \frac{2304}{c^2} W_{-3} W_{-2} +$ $+ \frac{180}{c} W_{-3} L_{-1}^2 + \frac{3312}{c^2} W_{-2}^2 L_{-1} + \frac{120}{c} W_{-2} L_{-1}^3\Big) v$

Conjecture 3.20 *Let $u \in V(c, \frac{1-r}{2}, 0)$ a subsingular vector. Then L_{-k} for $k > 1$ does not occur as a factor in u. Specially, $W_0 u, W_{-1} u \in J'(c, \frac{1-r}{2}, 0)$.*

Subsingular vector u in $V(-8h_W, h_W + \frac{5}{4}, h_W)$ ($p = r = 2$ case):

$$\begin{aligned}\Bigg(L_{-2}^2 &- \frac{3}{4h_W} W_{-4} - \left(\frac{3}{2h_W^2} + \frac{3}{2h_W}\right) W_{-3} W_{-1} + \frac{3}{2h_W} W_{-3} L_{-1} - \frac{3}{2h_W} W_{-1} L_{-3} \\ &- \frac{3}{h_W} W_{-1} L_{-2} L_{-1} + \left(\frac{3}{2h_W} + \frac{39}{4h_W^2}\right) W_{-1}^2 L_{-2} + \frac{9}{4h_W^2} W_{-1}^2 L_{-1}^2 + \\ &- \left(\frac{9}{4h_W^2} + \frac{117}{8h_W^3}\right) W_{-1}^3 L_{-1} + \left(\frac{135}{32h_W^4} + \frac{153}{32h_W^3} + \frac{9}{8h_W^2}\right) W_{-1}^4\Bigg) v\end{aligned}$$

Based on these examples, we state

Conjecture 3.21 *Suppose $2h_W + \frac{p^2-1}{12} c = 0$ for some $p \in \mathbb{N}$. Then $L'(c, h, h_W)$ is reducible if and only if $h = h_W + \frac{(13p+1)(p-1)}{12} + \frac{(1-r)p}{2}$.*

Remark 3.22 *Let* $2h_W + \frac{p^2-1}{12}c = 0$. *Then* $U(\mathcal{L})u' = V(c, h+p, h_W) \subseteq V(c, h, h_W)$. *However,* $V(c, h+p, h_W)$ *also contains a singular vector* $(u')^2$ *such that* $U(\mathcal{L})(u')^2 = V(c, h+2p, h_W)$ *and so on. Therefore,*

$$\cdots \supseteq V(c, h, h_W) \supseteq V(c, h+p, h_W) \supseteq \cdots \supseteq V(c, h+kp, h_W) \supseteq \cdots$$

If $h \neq h_W + \frac{(13p+1)(p-1)}{12} + \frac{(1-r)p}{2}$ *for all* $n \in \mathbb{Z}$, *then all the subquotients* $V(c, h+kp, h_W)/V(c, h+(k+1)p, h_W)$ *are irreducible modules* $L(c, h+kp, h_W)$.

4 Vertex operator algebra associated to $W(2,2)$ and intertwining operators

Let $c \neq 0$. It was shown in [ZD] that $L(c, 0, 0)$ is the only quotient of $V(c, 0, 0)$ with the structure of a vertex operator algebra (VOA). This algebra is always irrational, and all of its irreducible representations are known:

Theorem 4.1 ([ZD]) *Let* $c \neq 0$. *Then*

1. *There is a unique VOA structure on* $L(c, 0, 0)$ *with the vacuum vector* v, *and the Virasoro element* $\omega = L_{-2}v$. $L(c, 0, 0)$ *is generated with* ω *and* $x = W_{-2}v$ *and* $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, $Y(W_{-2}v, z) = \sum_{n\in\mathbb{Z}} W_n z^{-n-2}$.
2. *Any quotient module of* $V(c, h, h_W)$ *is an* $L(c, 0, 0)$*-module, and* $\{L(c, h, h_W) : h, h_W \in \mathbb{C}\}$ *gives a complete list of irreducible* $L(c, 0, 0)$*-modules up to isomorphism.*

Now we present a realization of intermediate series via intertwining operators for $L(c, 0, 0)$-modules.

Definition Let M_i, $i = 1, 2, 3$ modules over VOA $V = (V, Y, \mathbf{1})$. Linear map $\mathcal{I} : M_1 \otimes M_2 \to M_3\{w\} = \left\{\sum_{n\in\mathbb{Q}} u_n z^n : u_n \in M_k\right\}$, or, equivalently,

$$\mathcal{I} : M_1 \to (\mathrm{Hom}\,(M_2, M_3))\,\{z\}$$
$$\mathcal{I}\,(u, z) = \sum_{n\in\mathbb{Q}} u_n z^{-n-1}, \text{ with } u_n \in \mathrm{Hom}\,(M_j, M_k)\,,$$

is called an intertwining operator of type $\binom{M_3}{M_1\ M_2}$ if it satisfies:

i *Truncation property* - For any $u \in M_1$, $v \in M_2$, $u_n v = 0$ for $n >> 0$;

ii L_{-1}*-derivative property* - $\mathcal{I}(L_{-1}w, z) = \frac{d}{dx}\mathcal{I}(w, z)$ for any $w \in M_1$;

iii *Jacobi identity* - for any $a \in V$, $u \in M_1$ and $v \in M_2$

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(a, z_1)\mathcal{I}(u, z_2)v - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\mathcal{I}(u, z_2)Y(a, z_1)v$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\mathcal{I}(Y(a, z_0)u, z_2)v$$

Let $M(c,h,h_W)$ denotes a highest weight module. Suppose a nontrivial intertwining operator $\mathcal{I}$ of type $\binom{M(c,h_3 h_W')}{L'(c,h_1,0)\quad M(c,h_2,h_W)}$ exists. Let $h_1 \neq 0$ and $v \in L'(c,h,0)$ the highest weight vector. Let $I(v,z) = z^{-\alpha}\sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1}$ for $\alpha = h_1 + h_2 - h_3$. Recall that $W_0 v = W_{-1} v = 0$. Then

$$
\begin{aligned}
\left[L_m, v_{(n)}\right] &= \sum_{i\geq 0} \binom{m+1}{i} \left(L_{i-1}v\right)_{(m+n-i+1)} = \\
&= \left(L_{-1}v\right)_{(m+n+1)} + (m+1)\left(L_0 v\right)_{(m+n)} = \\
&= -\left(\alpha+n+m+1\right) v_{(m+n)} + (m+1)\, h_1 v_{(m+n)} = \\
&= -\left(n+\alpha+(1+m)(1-h_1)\right) v_{(m+n)}
\end{aligned}
$$

and

$$
\begin{aligned}
\left[W_m, v_{(n)}\right] &= \sum_{i\geq 0} \binom{m+1}{i} \left(W_{i-1}v\right)_{(m+n-i+1)} = \\
&= \left(W_{-1}v\right)_{(m+n+1)} + (m+1)\left(W_0 v\right)_{(m+n)} = 0
\end{aligned}
$$

so components $v_{(n)}$ span a module from intermediate series $V'_{\alpha,\beta,0}$ for $\beta = 1-h_1$. We have a nontrivial $\mathcal{L}$-operator

$$
\Phi : V'_{\alpha,\beta,0} \otimes M(c,h_2,h_W) \to M(c,h_3,h'_W), \quad \Phi(v_{(n)} \otimes x) = v_{(n)} x.
$$

Since dimensions of weight subspaces are infinite in $V'_{\alpha,\beta,0} \otimes M(c,h_2,h_W)$ and finite in $M(c,h_3,h'_W)$, we conclude that $V'_{\alpha,\beta,0} \otimes M(c,h_2,h_W)$ is reducible.

Let us mark some intertwining operators. Since $M(c,h,h_W)$ is $L(c,0,0)$-module, there exist intertwining operators of type $\binom{M(c,h,h_W)}{L(c,0,0)\quad M(c,h,h_W)}$, and transposed operators $\binom{M(c,h,h_W)}{M(c,h,h_W)\quad L(c,0,0)}$. In particular, operators of type

$$
\binom{L(c,h,0)}{L(c,h,0)\quad L(c,0,0)} \text{ and } \binom{L'(c,h,0)}{L'(c,h,0)\quad L(c,0,0)}
$$

exist.

5 Irreducibility of a module $V'_{\alpha,\beta,0} \otimes L(c,h,h_W)$

We state the main result of this section immediately:

Theorem 5.1 *Module $V'_{\alpha,\beta,0} \otimes L(c,h,h_W)$ is irreducible if and only if there is a subsingular vector u in $V(c,h,h_W)$, such that $\overline{u} = L^r_{-p} v$, and if $\alpha + (1-p)\beta \notin \mathbb{Z}$.*

We shall prove this theorem in several steps, using analogous approach as in Virasoro case (see [R2]). First we show that some relation in $L(c,h,h_W)$ is needed to obtain irreducibility.

Theorem 5.2 *Module $V'_{\alpha,\beta,0} \otimes V(c,h,h_W)$ is reducible.*

Proof. Let v the highest weight vector in $V(c,h,h_W)$. We show $v_k \otimes v$ generates a proper submodule, for any $k \in \mathbb{Z}$. Assume otherwise. Then w exists in $U(\mathcal{L})$, such that $w(v_k \otimes v) = v_{k-1} \otimes v$. Since $U(\mathcal{L}_+)v = 0$, we have $v_{k-1} \otimes v = \sum_{i=0}^{n} w_{i+1}(v_{k+i} \otimes v)$ for some $0 \neq w_j \in U(\mathcal{L}_-)_{-j}$. But then $v_{k+n} \otimes w_{n+1}v = 0$ which means $w_{n+1}v = 0$. This is impossible, since Verma module is free over $U(\mathcal{L}_-)$. ■

Next we prove irreducibility criterion analogous to one proved in [Zh], and generalized in [R2].

Theorem 5.3 *Module $V'_{\alpha,\beta,0} \otimes L(c,h,h_W)$ is irreducible if and only if it is cyclic on every vector $v_k \otimes v$.*

Proof. We follow the proof of Theorem 3.2 in [R2]. The only if part is trivial. Assume $V'_{\alpha,\beta,0} \otimes L(c,h,h_W)$ is cyclic on every $v_k \otimes v$. Let U a submodule and $0 \neq x \in U$ homogeneous vector. Then

$$x = v_{m-n} \otimes x_0 + \cdots v_m \otimes x_n$$

for some $x_j \in V(c,h,h_W)_j$. We use induction on n to show there is $v_k \otimes v \in U$ for some $k \in \mathbb{Z}$. If $n = 0$, $x_n \in \mathbb{C}v$ and we are done. Let $n > 0$. Our strategy is to act on x by $U(\mathcal{L})$ in order to 'shorten it' and apply induction. Recall $W_k v_i = 0$ for any k, i. If $W_k x_n \neq 0$ for $k \in \mathbb{N}$, we have

$$W_k x = v_{m-n+k} \otimes y_0 + \cdots + v_m \otimes y_{n-k} \neq 0$$

where $y_j = W_k x_{j+k} \in V(c,h,h_W)_j$. By inductive hypothesis, now there must be some $v_k \otimes v \in U$. Assume, therefore, that $W_k x_n = 0$ for all $k \in \mathbb{N}$. Since W_1, W_2, L_1 and L_2 generate $U(\mathcal{L}_+)$, vectors $L_1 x_n$ and $L_2 x_n$ can not both equal zero, for otherwise, x_n would be a singular vector in $L(c,h,h_W)$ other than v. But now we can follow the proof of Theorem 3.2 in [R2]. This completes the proof. ■

Define $U_n := U(\mathrm{Vir})(v_n \otimes v)$. Proof of Theorem 5.3 actually shows

Corollary 5.4 *Let M a nontrivial submodule in $V'_{\alpha,\beta,0} \otimes L(c,h,h_W)$. Then M contains U_n for some $n \in \mathbb{Z}$.*

Module $V'_{\alpha,\beta,0} \otimes L(c,h,h_W)$ is irreducible if and only if $U_n = U_{n+1}$ for all $n \in \mathbb{Z}$. Since $L_1(v_n \otimes v) = -(n+\alpha+2\beta) = v_{n+1} \otimes v$, we get $U_n \supseteq U_{n+1}$ for $n \neq -\alpha - 2\beta$. To obtain the other inclusion, we need a subsingular vector.

Lemma 5.5 *Let $k = k_1 + \cdots k_s$ for $k_i \in \mathbb{N}$. Then*

$$L_{-k_s} \cdots L_{-k_1}(v_n \otimes v) = \sum_{i=0}^{k} u_i (v_{n-i} \otimes v)$$

for some $u_i \in U(\mathrm{Vir}_-)$.

Proof. From (1) we get

$$L_{-k_s}\cdots L_{-k_1}(v_n\otimes v)=\sum_{i=0}^{k} v_{n-i}\otimes L_{-j_m}\cdots L_{-j_1}v$$

for $0\le j_1+\cdots+j_m=k-i$. Next we show

$$v_{n-i}\otimes L_{-j_m}\cdots L_{-j_1}v=\sum_{j=0}^{k-i} x_j(v_{n-k+j}\otimes v) \tag{4}$$

for some $x_j\in U(\mathcal{L}_-)_{-j}$. But since

$$\begin{aligned} v_{n-i}\otimes L_{-j_m}\cdots L_{-j_1}v &= L_{-j_m}(v_{n-i}\otimes L_{-j_{m-1}}\cdots L_{-j_1}v)+ \\ &+(n-i+\alpha+\beta-j_m\beta)v_{n-i-j_m}\otimes L_{-j_{m-1}}\cdots L_{-j_1}v \end{aligned}$$

we may use induction on m to complete the proof. ∎

Theorem 5.6 *Let $uv\in(c,h,h_W)$ a subsingular vector such that $\overline{u}=L_{-p}^r$. If $\alpha+(1-p)\beta\notin\mathbb{Z}$, module $V'_{\alpha,\beta,0}\otimes L(c,h,h_W)$ is irreducible.*

Proof. For $n\in\mathbb{N}$, we need to find $x\in U(\mathcal{L})$ such that $x(v_n\otimes v)=v_{n-1}\otimes v$. Since $uv=L_{-p}^r v+\sum w_i L_{-k_s}\cdots L_{-k_1}v=0$, we have

$$u(v_{n+rp-1}\otimes v)=L_{-p}^r(v_{n+rp-1}\otimes v)+\sum w_i L_{-k_s}\cdots L_{-k_1}(v_{n+rp-1}\otimes v)$$

where $w_i\in\mathcal{W}_{-i}$, $\deg_W w_i>0$ and $0<k<rp$ for $k=k_1+\cdots+k_s$. From Lemma 5.5 and the fact that $w(v_k\otimes x)=v_k\otimes wx$, for $w\in\mathcal{W}$, we get $y_i\in U(\mathcal{L}_-)_{rp-i}$ such that

$$\sum_{i=0}^{rp-1} y_i(v_{n+rp-1-i}\otimes v)=L_{-p}^r(v_{n+rp-1}\otimes v). \tag{5}$$

Note that $y_0=u$. Set $\lambda_j:=(n+(r-j)p-1+\alpha+(1-p)\beta)$, and $\Lambda_i=\prod_{j=0}^{r-i-1}\lambda_j$. Then (5) becomes

$$\sum_{i=0}^{rp-1} y_i(v_{n+rp-1-i}\otimes v)=\sum_{i=0}^{r-1}(-1)^{r-i}\binom{r}{i}\Lambda_i v_{n+ip-1}\otimes L_{-p}^i v. \tag{6}$$

Next we want to eliminate components on the right side of (6), starting with $v_{n+(r-1)p-1}\otimes L_{-p}^{r-1}v$. Using induction, one can show

$$-\binom{r}{r-1}\lambda_0 L_{-p}^{r-1}(v_{n+(r-1)p-1}\otimes v)= \tag{7}$$

$$=-\binom{r}{r-1}\lambda_0\sum_{i=0}^{r-1}(-1)^{r-1-i}\binom{r-1}{i}\left(\prod_{j=1}^{r-1-i}\lambda_j\right)v_{n+ip-1}\otimes L_{-p}^i v=$$

$$=\sum_{i=0}^{r-1}(-1)^{r-i}\binom{r}{r-1}\binom{r-1}{i}\Lambda_i v_{n+ip-1}\otimes L_{-p}^i v.$$

The right side of (6)-(7) equals

$$\sum_{i=0}^{r-2}(-1)^{r-i}\left(\binom{r}{i}-\binom{r}{r-1}\binom{r-1}{i}\right)\Lambda_i v_{n+ip-1}\otimes L_{-p}^{i}v. \tag{8}$$

Coefficient with Λ_{r-2} is $\binom{r}{r-2}-\binom{r}{r-1}\binom{r-1}{r-2}=-\binom{r}{r-2}$ so we use

$$-\binom{r}{r-2}\lambda_0\lambda_1 L_{-p}^{r-2}(v_{n+(r-2)p-1}\otimes v)= \tag{9}$$

$$=-\binom{r}{r-2}\lambda_0\lambda_1\sum_{i=0}^{r-2}(-1)^{r-2-i}\binom{r-2}{i}\left(\prod_{j=2}^{r-1-i}\lambda_j\right)v_{n+ip-1}\otimes L_{-p}^{i}=$$

$$=-\sum_{i=0}^{r-2}(-1)^{r-i}\binom{r}{r-2}\binom{r-2}{i}\Lambda_i v_{n-(r-i)p-1}\otimes L_{-p}^{i}$$

and the right side of (8)-(9) is

$$\sum_{i=0}^{r-3}(-1)^{r-i}\left(\binom{r}{i}-\binom{r}{r-1}\binom{r-1}{i}+\binom{r}{r-2}\binom{r-2}{i}\right)\Lambda_i v_{n+ip-1}\otimes L_{-p}^{i}v.$$

In $r-1$ steps we get

$$(-1)^r\binom{r}{0}\left(\sum_{j=0}^{r-1}(-1)^j\binom{r}{r-j}\binom{r-j}{0}\right)\Lambda_0 v_{n-1}\otimes v=-\left(\prod_{j=0}^{r-1}\lambda_j\right)v_{n-1}\otimes v=$$

$$=-\left(\prod_{j=0}^{r-1}(n+(r-j)p-1+\alpha+(1-p)\beta)\right)v_{n-1}\otimes v.$$

Therefore, $x_i\in U(\mathcal{L})$ exist, such that

$$\sum_{i=0}^{rp-1}x_i(v_{n+rp-1-i}\otimes v)=\left(\prod_{j=0}^{r-1}(n+(r-j)p-1+\alpha+(1-p)\beta)\right)v_{n-1}\otimes v.$$

From condition $\alpha+(1-p)\beta\notin\mathbb{Z}$, follows

$$U_{n-1}\subseteq U_n+\cdots+U_{n+rp-1}. \tag{10}$$

If $\alpha+2\beta\notin\mathbb{Z}$ we have $U_n\supseteq U_{n+1}$ so (10) becomes $U_{n-1}=U_n$. Assume $\alpha+2\beta=-k$. Then using L_1 and L_2 we get

$$\cdots\supseteq U_{k-1}\supseteq U_k, U_{k+1}\supseteq U_{k+2}\supseteq\cdots$$

and (10) shows $U_n\subseteq U_{n+1}$ for $n\neq k-1$, $U_{k-1}\subseteq U_k+U_{k+1}$. But then

$$U_{k+2}\subseteq U_k\subseteq U_{k+1}=U_{k+2}$$

shows $U_k=U_{k+1}$ and so $U_n=U_{n+1}$ for all $n\in\mathbb{Z}$. This completes the proof. ■

Theorem 5.7 *If a singular vector $u' \in \mathcal{W}$ generates $J(c,h,h_W)$, module $V'_{\alpha,\beta,0} \otimes L(c,h,h_W)$ is reducible.*

Proof. Just like in Theorem 5.2, we show $U_n \neq U_{n-1}$. If $x \in U(\mathcal{L})$ exists, such that $x(v_n \otimes v) = v_{n-1} \otimes v$, then we can find $x_i \in U(\mathcal{L})_{-i}$ such that $v_{n-1} \otimes v = \sum_{i=0}^{k} x_{i+1}(v_{n+i} \otimes v)$. Than $v_{n+k} \otimes x_{k+1} v = 0$ so $x_{k+1}v = 0$ meaning $x \in U(\mathcal{L})u'$. But since $u' \in \mathcal{W}$, we have $x_{k+1}(v_{n+k} \otimes v) = 0$ so this component of the sum is useless. Repeating the process we get a contradiction. ■

Theorem 5.8 *Let $uv \in (c,h,h_W)$ a subsingular vector such that $\overline{u} = L_{-p}^r$. If $\alpha + (1-p)\beta \in \mathbb{Z}$, module $V'_{\alpha,\beta,0} \otimes L(c,h,h_W)$ is reducible. There exists $k \in \mathbb{Z}$ such that U_k is irreducible.*

Proof. Since α is invariant modulo $\mathbb{Z}$ we may assume $\alpha + (1-p)\beta = 0$. From the proof of Theorem 5.6 we have

$$x(v_n \otimes v) = -\left(\prod_{j=0}^{r-1}(n-1+(r-j)p)\right) v_{n-1} \otimes v$$

for some $x \in U(\mathcal{L})$, so for $n \notin \{1-p, 1-2p, \ldots, 1-rp\}$ we get $U_n = U_{n+1}$. Now we prove $U_{1-jp} \neq U_{-jp}$ for $j = 1, \ldots, r$. We use the same reasoning as in previous proofs. Assume otherwise. Then $y \in U(\mathcal{L})$ exists, such that $y(v_{1-jp} \otimes v) = v_{-jp} \otimes v$, i.e. there are $y_i \in U(\mathcal{L}_-)_{-i}$ such that

$$v_{-jp} \otimes v = \sum_{i=0}^{m-1} y_{i+1}(v_{1-jp+i} \otimes v). \tag{11}$$

But then $y_m v = 0$. We may assume $y_m \in U(\mathcal{L}_-)u$ because $u'(v_{m-jp} \otimes v) = 0$. Let $y_m = zu$, $z \in U(\mathcal{L}_-)_{-(m-rp)}$. Then

$$y_m(v_{m-jp} \otimes v) = z\left(L_{-p}^r + \sum w_i L_{-k_s} \cdots L_{-k_1}\right)(v_{m-jp} \otimes v).$$

On the right side, we get sum of $z(v_{m-jp-i} \otimes x_i v)$ where $i = 0, 1, \ldots rp$. However, since $uv = 0$, all the components $v_{m-jp} \otimes x_0 v$ add to zero. Like in Lemma 5.5 we get $z_i \in U(\mathcal{L})$ such that

$$y_m(v_{m-jp} \otimes v) = \sum_{i=1}^{rp} z_i(v_{m-i-jp} \otimes v).$$

However, this means that $y_m(v_{m-jp} \otimes v)$ is unnecessary in (11) since it can be expressed with $v_{m-1-jp} \otimes v, \ldots, v_{m-rp-jp} \otimes v$ (for some $m \geq rp$). Repeating the process, we get to $y_{rp} = u$. However,

$$u(v_{1-jp} \otimes v) = \left(\sum w_i L_{-k_s} \cdots L_{-k_1}\right)(v_{1-jp} \otimes v) \tag{12}$$

because $L_{-p}^r(v_{1-jp} \otimes v) = 0$. Obviously, the right side of (12) can not produce $v_{-jp} \otimes v$, leading to contradiction. Therefore, $U_{1-jp} \subsetneq U_{-jp}$.

Module U_{1-p} is irreducible by Corollary 5.4. ■

Theorems 5.2, 5.7, 5.8 and 5.6 combined prove Theorem 5.1.

Note that U_{1-p} in Theorem 5.8 is irreducible module with infinite-dimensional weight subspaces.

Corollary 5.9 *Module $V'_{\alpha,\beta,0} \otimes L(c,h,h_W)$ contains an irreducible (not necessarily proper) submodule with infinite-dimensional weight subspaces if and only if, a subsingular vector u, such that $\overline{u} = L_{-p}^r v$ exists in $V(c,h,h_W)$.*

6 Subquotients of a module $V'_{\alpha,\beta,0} \otimes L(c,h,h_W)$

Remark 6.1 *Let U_n/U_{n+1} a nontrivial subquotient in $V'_{\alpha,\beta,0} \otimes L(c,h,h_W)$. Then*

$$L_k(v_n \otimes v) = \lambda v_{n+k} \otimes v \in U_{n+k} \subseteq U_{n+1}, \text{ for } k > 0$$

$$L_0(v_n \otimes v) = (h - n - \alpha - \beta) v_n \otimes v$$
$$W_0(v_n \otimes v) = v_n \otimes W_0 v = h_W v_n \otimes v$$

shows U_n/U_{n+1} is the highest weight module with the highest weight (c, h_n, h_W), where $h_n = h - n - \alpha - \beta$. From Theorem 2.1 follows that $V(c,h_n,h_W)$ is reducible if and only if $V(c,h,h_W)$ is reducible. If that is the case, formula for a singular vector $u' \in \mathcal{W}$ is the same in both modules, so $u'(v_n \otimes v) = 0$. This shows U_n/U_{n+1} is isomorphic to either to $L'(c,h_n,h_W)$, or to $L(c,h_n,h_W)$ (see Corollary 3.19). Of course, if $h_n \neq h_W \frac{(13p+1)(p-1)}{12} + \frac{(1-r)p}{2}$ for all $r \in \mathbb{N}$, these two modules are the same, $L(c,h_n,h_W)$. Otherwise, the question is weather $u(v_n \otimes v) \in U_{n+1}$.

Let us review subquotients found in Theorems 5.2, 5.7, and 5.8.

Theorem 6.2 *Verma modules $V(c, h-\alpha-\beta-n, h_W)$ occur as subquotients in $V'_{\alpha,\beta,0} \otimes V(c,h,h_W)$ for any $n \in \mathbb{Z}$, with exception of $n = -\alpha$ if $\alpha \in \mathbb{Z}$ and $\beta = 0$, and $n = -\alpha - 1$ if $\alpha \in \mathbb{Z}$ and $\beta = 1$.*

Proof. For any $n \in \mathbb{Z}$ (with noted exceptions), U_n/U_{n+1} is the highest weight module with the weight $(c, h-\alpha-\beta-n, h_W)$. (If $\alpha + 2\beta + n = 0$ then consider U_n/U_{n+2} instead.) Let us prove, this module is free over $U(\mathcal{L}_-)$. Note that

$$U_{n+1} = \operatorname{span}\{x\,(v_{n+k} \otimes v) : x \in U\,(\mathcal{L}_-)\,,\ k > 0\}$$

and every $x\,(v_{n+k} \otimes v)$ has a component $v_{n+k} \otimes xv \neq 0$ since Verma module is free over $U\,(\mathcal{L}_-)$. Suppose U_n/U_{n+1} is not free. Then $y \in U(\mathcal{L}_-)$ exists, such that $y(v_n \otimes v) \in U_{n+1}$. But $y(v_n \otimes v)$ can not have a component $v_{n+k} \otimes xv$ for $k > 0$, proving a contradiction. ■

Theorem 6.3 *If a singular vector* $u' \in \mathcal{W}$ *generates* $J(c,h,h_W)$*, then modules* $L(c,h-\alpha-\beta-n,h_W)$ *occur as subquotients in* $V'_{\alpha,\beta,0}\otimes L(c,h,h_W)$ *for any* $n\in\mathbb{Z}$*, with exception of* $n=-\alpha$ *if* $\alpha\in\mathbb{Z}$ *and* $\beta=0$*, and* $n=-\alpha-1$ *if* $\alpha\in\mathbb{Z}$ *and* $\beta=1$*.*

Proof. Follows from Theorem 5.7, Remark 6.1 and the fact that $L(c,h-\alpha-\beta-n,h_W)$ is a quotient of $L'(c,h-\alpha-\beta-n,h_W)$. ■

Theorem 6.4 *If there is a subsingular vector* $u\in V(c,h,h_W)$ *such that* $\overline{u}=L_{-p}^r v$*, and* $\alpha+(1-p)\beta\in\mathbb{Z}$*, then* $L(c,h+(j-\beta)p,h_W)$ *occur as subquotients in* $V'_{\alpha,\beta,0}\otimes L(c,h,h_W)$ *for* $j=1,\dots,r$*, with the exception of* $L(c,h,0)$ *in* $V'_{0,1,0}\otimes L(c,h,0)$*.*

Proof. Follows from Remark 6.1 and Theorem 5.8. ■

Remark 6.5 *We expect existence of intertwining operators of type*

$$\binom{L(c,h+(r-\beta)p,h_W)}{L(c,1-\beta,0)\quad L(c,h,h_W)}$$

what would prove Theorems 5.8 and 6.4.

Corollary 6.6 $V'_{\alpha,\beta,0}\otimes L(c,0,0)$ *is irreducible if and only if* $\alpha\notin\mathbb{Z}$*. If* $2\beta-1\notin\mathbb{N}$*, then*

$$\left(V'_{0,\beta,0}\otimes L(c,0,0)\right)/U_1\cong L(c,1-\beta,0).$$

Proof. (Sub)singular vector in $L(c,0,0)$ is $L_{-1}v$ so we apply Theorem 6.4. ■

Let $2\beta-1=r'\in\mathbb{N}$. Then $\left(V'_{0,\beta,0}\otimes L(c,0,0)\right)/U_0$ is the highest weight module with the highest weight $(c,\frac{1-r'}{2},0)$. Assume there is a subsingular vector $u\in V(c,\frac{1-r'}{2},0)$ (see Conjecture 3.21), such that $\overline{u}=L_{-1}^{r'}v$ and that L_k does not occur as a factor in u, for $k>1$ (see Conjecture 3.20). Then $u=(L_{-1}^{r'}+\sum_{i=0}^{r'-1}w_iL_{-1}^i)v$ for some $w_i\in\mathcal{W}$ so

$$u(v_{-1}\otimes v)=\sum_{i=0}^{r'}(-1)^i i! v_{-1-i}\otimes w_iv.$$

Since $L_{-1}(v_0\otimes v)=W_{-1}(v_0\otimes v)=0$, we get

$$U_0=\operatorname{span}\left\{x(v_k\otimes v):k\in\mathbb{N},x\in U(\mathcal{L}_-\setminus\{L_{-1},W_{-1}\}\right\},$$

and every $x(v_k\otimes v)$ has a component $v_k\otimes xv\neq 0$. Then, obviously, $u'(v_{-1}\otimes v)\notin U_0$ so

$$\left(V'_{0,\frac{1+r'}{2},0}\otimes L(c,0,0)\right)/U_0\cong L'(c,\frac{1-r'}{2},0).$$

Remark 6.7 *Since intertwining operators of types*

$$\binom{L(c,1-\beta,0)}{L(c,1-\beta,0) \quad L(c,0,0)}, \ \textit{and} \ \binom{L'(c,\frac{1-r'}{2},0)}{L'(c,\frac{1-r'}{2},0) \quad L(c,0,0)}$$

exist, there are nontrivial $\mathcal{L}$-homomorphisms

$$\begin{aligned} V'_{0,\beta,0} \otimes L(c,0,0) &\to L(c,1-\beta,0), \\ V'_{0,\frac{1+r'}{2},0} \otimes L(c,0,0) &\to L'(c,\frac{1-r'}{2},0). \end{aligned}$$

7 Twisted Heisenberg-Virasoro algebra

The twisted Heisenberg-Virasoro algebra is the universal central extension of the Lie algebra of differential operators on a circle of order at most one:

$$\left\{ f(t)\frac{d}{dt} + g(t) : f,g \in \mathbb{C}[t,t^{-1}] \right\}.$$

It has an infinite-dimensional Heisenberg subalgebra, and a Virasoro subalgebra. Its highest weight representations have been studied in [Ar] and [B]. In this paper we focus on zero level case (trivial action of central element of the Heisenberg subalgebra), studied in [B]. These representations occur in the construction of modules for the toroidal Lie algebras [B2]. Our goal is to find irreducible representations with infinite-dimensional weight spaces. As is case with $W(2,2)$, we study tensor product of an intermediate series and the highest weight module, and use singular vectors in Verma modules to check irreducibility. However, there are no subsingular vectors in Verma modules over the Heisenberg-Virasoro algebra. Still, results are quite similar to those for $W(2,2)$ algebra when Heisenberg subalgebra acts trivially on an intermediate series. We omit some details since most proofs are analogous to $W(2,2)$ case.

Twisted Heisenberg-Virasoro algebra $\mathcal{H}$ is a Lie algebra with basis

$$\{L_n, I_n : n \in \mathbb{Z}\} \cup \{C_L, C_{LI}, C_I\}$$

and Lie bracket

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}C_L, \\ [L_n, I_m] &= -mI_{n+m} - \delta_{n,-m}(n^2+n)C_{LI}, \\ [I_n, I_m] &= n\delta_{n,-m}C_I, \\ [\mathcal{H}, C_L] &= [\mathcal{H}, C_{LI}] = [\mathcal{H}, C_I] = 0. \end{aligned}$$

Obviously, $\{L_n, C_L : n \in \mathbb{Z}\}$ spans a Virasoro subalgebra, and $\{I_n, C_I : n \in \mathbb{Z}\}$ spans a Heisenberg subalgebra. Center of $\mathcal{H}$ is spanned by $\{I_0, C_L, C_I, C_{LI}\}$

and, unlike $W(0)$ in $W(2,2)$, $I(0)$ acts semisimply on $\mathcal{H}$. Standard $\mathbb{Z}$-gradation

$$\mathcal{H}_n = \mathbb{C}L_n \oplus \mathbb{C}I_n, \quad n \in \mathbb{Z}^*,$$
$$\mathcal{H}_0 = \mathbb{C}L_0 \oplus \mathbb{C}I_0 \oplus \mathbb{C}C_L \oplus \mathbb{C}C_I \oplus \mathbb{C}C_{LI}$$

induces a triangular decomposition

$$\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_0 \oplus \mathcal{H}_+$$

as usual.

Let $U(\mathcal{H})$ a universal enveloping algebra. For arbitrary $h, h_I, c_L, c_I, c_{LI} \in \mathbb{C}$, let $\mathcal{I}$ a left ideal in $U(\mathcal{H})$ generated by $\{L_n, I_n, L_0 - h\mathbf{1}, I_0 - h_I\mathbf{1}, C_L - c_L\mathbf{1}, C_I - c_I\mathbf{1}, C_{LI} - c_{LI}\mathbf{1} : n \in \mathbb{N}\}$. Then $U(\mathcal{H})/\mathcal{I}$ is a Verma module denoted with $V(c_L, c_I, c_{LI}, h, h_I)$. As usual, it is a free $U(\mathcal{H}_-)$-module generated by the highest weight vector $v := \mathbf{1} + \mathcal{I}$, and a standard PBW basis

$$\{I_{-m_s} \cdots I_{-m_1} L_{-n_t} \cdots L_{-n_1} v : m_s \geq \cdots \geq m_1 \geq 1, n_t \geq \cdots \geq n_1 \geq 1\}.$$

It has a unique maximal submodule $J(c_L, c_I, c_{LI}, h, h_I)$, and a quotient module $L(c_L, c_I, c_{LI}, h, h_I) = V(c_L, c_I, c_{LI}, h, h_I)/C(c_L, c_I, c_{LI}, h, h_I)$ is irreducible. In this paper we focus on zero level highest weight representations, i.e. $c_I = 0$ case.

Theorem 7.1 ([B]) *Let $c_I = 0$ and $c_{LI} \neq 0$.*

(i) *If $\frac{h_I}{c_{LI}} \notin \mathbb{Z}$ or $\frac{h_I}{c_{LI}} = 1$, then Verma module $V(c_L, 0, c_{LI}, h, h_I)$ is irreducible.*

(ii) *If $\frac{h_I}{c_{LI}} \in \mathbb{Z} \setminus \{1\}$ then $V(c_L, 0, c_{LI}, h, h_I)$ possesses a singular vector $u \in V_p$, where $p = |\frac{h_I}{c_{LI}} - 1|$. In this case, a quotient module $L(c_L, 0, c_{LI}, h, h_I) = V(c_L, 0, c_{LI}, h, h_I)/U(\mathcal{H}_-)u$ is irreducible.*

Define I-degree as follows:

$$\deg_I L_{-n} = 0, \quad \deg_I I_{-n} = 1,$$

which induces $\mathbb{Z}$-grading on $U(\mathcal{H})$ and on V

$$\deg_I I_{-m_s} \cdots I_{-m_1} L_{-n_t} \cdots L_{-n_1} v = s.$$

For a non-zero $x \in V$ given in standard PBW basis we denote by $\overline{x}$ its lowest non-zero homogeneous component with respect to I-degree.

Also, let $\mathcal{I} = \{I_{-m_s} \cdots I_{-m_1} v : m_s \geq \cdots \geq m_1 \geq 1\}$. By abuse of notation, we will sometimes consider $\mathcal{I}$ as a subalgebra of $U(\mathcal{H}_-)$. We write V short for $V(c_L, 0, c_{LI}, h, h_I)$.

Theorem 7.2 ([B]) *Let $u \in V_p$ a singular vector. If $1 - \frac{h_I}{c_{LI}} = p \in \mathbb{N}$, then $\overline{u} = L_{-p}v$. If $\frac{h_I}{c_{LI}} - 1 = p \in \mathbb{N}$ then $\overline{u} = I_{-p}v$ and $u \in V_p \cap \mathcal{I}$.*

Remark 7.3 *If $\frac{h_I}{c_{LI}} - 1 = p \in \mathbb{N}$, one can show that*

$$u = w_0 v + \sum_{i=1}^{p-1} w_i L_{-i} v + L_{-p} v, \tag{13}$$

where $w_i \in U(\mathcal{H}_-)_{p-i} \cap \mathcal{I}$. (Set $r = 1$ in Lemma 3.15.)

For example $(L_{-1} + \frac{h}{c_{LI}} I_{-1})v$ is a singular vector in $V(c_L, 0, c_{LI}, h, 0)$, and $I_{-1}v$ in $V(c_L, 0, c_{LI}, h, 2c_{LI})$.

Next we define an intermediate series. We take an intermediate series Vir-module $V_{\alpha,\beta}$ and let I_n act by scalar. Let $\alpha, \beta, F \in \mathbb{C}$. $V_{\alpha,\beta,F}$ is $\mathcal{H}$-module with basis $\{v_m : m \in \mathbb{Z}\}$ and action

$$\begin{aligned} L_n v_m &= -(m + \alpha + \beta + n\beta) v_{m+n}, \\ I_n v_m &= F v_{m+n}, \\ C_L v_m &= C_I v_m = C_{LI} v_m = 0. \end{aligned}$$

As with other intermediate series, $V_{\alpha,\beta,F} \cong V_{\alpha+k,\beta,F}$ for $k \in \mathbb{Z}$. Also, $V_{\alpha,\beta,F}$ is reducible if and only if $\alpha \in \mathbb{Z}$, $\beta \in \{0, 1\}$ and $F = 0$. Define $V'_{0,0,0} := V_{0,0,0}/\mathbb{C}v_0$, $V'_{0,1,0} := \oplus_{k \neq -1} \mathbb{C} v_k$ and $V'_{\alpha,\beta,F} := V_{\alpha,\beta,F}$ otherwise. Then $V'_{\alpha,\beta,F}$ are irreducible modules.

It has been shown in [LuZ] that an irreducible $\mathcal{H}$-module with finite-dimensional weight spaces is either the highest (or lowest) weight module, or isomorphic to some $V'_{\alpha,\beta,F}$.

Throughout this section we write short $M(\bar{c}, h, h_I)$ for highest weight module $M(c_L, 0, c_{LI}, h, h_I)$.

Now consider module $V'_{\alpha,\beta,F} \otimes L(\bar{c}, h, h_I)$. It is generated by $\{v_m \otimes v : m \in \mathbb{Z}\}$ where v is the highest weight vector, and has infinite-dimensional weight subspaces.

Theorem 7.4 *Let $\alpha, \beta, F \in \mathbb{C}$ arbitrary. $V'_{\alpha,\beta,F} \otimes L(\bar{c}, h, h_I)$ is irreducible if and only if it is cyclic on every $v_m \otimes v$, $m \in \mathbb{Z}$.*

Proof. The proof is similar to $W(2,2)$ case (Theorem 5.3). We take nontrivial submodule U,

$$x = v_{m-n} \otimes x_0 + \cdots v_m \otimes x_n \in U$$

such that $x_i \in L(\bar{c}, h, h_I)_i$, and show by induction on n that $v_k \otimes v \in U$. If $L_1 x_n \neq 0$ or $L_2 x_n \neq 0$, we follow proof of Theorem 3.2 in [R2]. Otherwise it must be $I_1 x_n \neq 0$.

If $F = 0$, then

$$I_1 x = v_{m-n+1} \otimes I_1 x_1 + \cdots + v_m \otimes I_1 x_n \neq 0$$

and the proof is done. If $F \neq 0$ we have

$$(I_1^2 - FI_2)x = 2F\sum_{i=1}^{n} v_{m-n+i+1} \otimes I_1 x_i + \sum_{i=2}^{n} v_{m-n+i} \otimes (I_1^2 - FI_2)x_i =$$

$$= Fv_{m+1} \otimes I_1 x_n + \sum_{i=0}^{n-2} v_{m-n+i+2} \otimes (2FI_1 x_{i+1} + (I_1^2 - FI_2)x_{i+2})$$

so again, there is a vector in U with less than $n+1$ components, and by induction, there is some $v_k \otimes v \in U$. ■

We denotation by U_k submodule $U(\mathcal{H})(v_k \otimes v)$ for any $k \in \mathbb{Z}$. In order to prove irreducibility of $V'_{\alpha,\beta,F} \otimes L(\overline{c}, h, h_I)$, it suffices to show $U_n = U_{n+1}$ for all n.

Theorem 7.5 *Module $V'_{\alpha,\beta,F} \otimes V(\overline{c}, h, h_I)$ is reducible. Modules $V(\overline{c}, h - \alpha - \beta - n, h_I + F)$ occur as subquotients, for any $n \in \mathbb{Z}$, with exception of $n = -\alpha$ if $\alpha \in \mathbb{Z}$ and $\beta = 0$, and $n = -\alpha - 1$ if $\alpha \in \mathbb{Z}$ and $\beta = 1$.*

Proof. Analogous to Theorems 5.2 and 6.2. ■

First we consider case $F = 0$.

Theorem 7.6 *Let $1 - \frac{h_I}{c_{LI}} = p \in \mathbb{N}$. Then $V'_{\alpha,\beta,0} \otimes L(\overline{c}, h, h_I)$ is reducible if and only if $\alpha + (1-p)\beta \in \mathbb{Z}$. In that case $(V'_{\alpha,\beta,0} \otimes L(\overline{c}, h, h_I))/U_{1-p}$ is the highest weight module with the highest weight $(\overline{c}, h + p(1-\beta), h_I)$.*

Proof. The proof of irreducibility is analogous to that of Theorem 5.6 for $r = 1$, since we have a singular vector (13). Proof of reducibility is analogous to Theorem 6.4 for $r = 1$. ■

Theorem 7.7 *Let $\frac{h_I}{c_{LI}} - 1 \in \mathbb{N}$. Then $V'_{\alpha,\beta,0} \otimes L(\overline{c}, h, h_I)$ is reducible. Subquotients U_n/U_{n+1} are isomorphic to $L(\overline{c}, h - \alpha - \beta - n, h_I)$.*

Proof. Analogous to Theorem 6.3. ■

Now we give a general result for arbitrary F, analogous to Theorem 5 in [Zh].

Theorem 7.8 *Let $\left|1 - \frac{h_I}{c_{LI}}\right| = p \in \mathbb{N}$, and $(h, h_I) \neq (0,0)$. If F is transcendental over $\mathbb{Q}(\alpha, \beta, c_L, c_{LI}, h, h_I)$ or F is algebraic over $\mathbb{Q}(\alpha, \beta, c_L, c_{LI}, h, h_I)$ with degree greater than p, then $V'_{\alpha,\beta,F} \otimes L(\overline{c}, h, h_I)$ is irreducible.*

Proof. Since $F \neq 0$ we have

$$I_1(v_{n-1} \otimes v) = Fv_n \otimes v \neq 0$$

so $U_n \subseteq U_{n-1}$.

Let $p = \frac{h_I}{c_{LI}} - 1$. Then $u \in \mathcal{I}$. Since

$$I_{-j_t} \cdots I_{-j_1}(v_{n+p-1} \otimes v) = F^t v_{n-1} \otimes v + \cdots + v_{n+p-1} \otimes I_{-j_t} \cdots I_{-j_1} v$$

for $j_1 + \cdots + j_t = p$, we find $u' \in U(\mathcal{H})$ such that

$$u'(v_n \otimes v) = Fs(F) v_{n-1} \otimes v$$

for some $s(F) \in \mathbb{Q}(h_I, c_{LI})\,[F]$, $\deg s = p - 1$. By assumption, $Fs(F) \neq 0$ so $U_{n-1} \subseteq U_n$.

Now let $p = 1 - \frac{h_I}{c_{LI}}$. Then we apply (13). Since

$$\begin{aligned} I_{-j_t} \cdots I_{-j_1} L_{-i}(v_{n+p} \otimes v) &= -F^t(n + p + \alpha + \beta - i\beta) v_n \otimes v + \\ &+ \sum_k F^k(n + p + \alpha + \beta - i\beta) \sum_j v_{n+p-j} \otimes x_j^{(t-k)} v + \\ &+ \sum_l F^l \sum_j v_{n+p-j-i} \otimes y_j^{(t-l)} L_{-i} v + v_{n+p} \otimes I_{-j_t} \cdots I_{-j_1} L_{-i} v \end{aligned}$$

where $j_1 + \cdots + j_t = p - i$ and $\deg_I x_j^{(r)} = \deg_I y_j^{(r)} = r$, we get

$$u(v_{n+p} \otimes v) = f(F) v_n \otimes v + \sum_{i=1}^{p-1} v_{n+i} \otimes z_i v$$

for some $z_i \in U(\mathcal{H}_-)_{-i}$ and polynomial $f(F)$. Then we find $u' \in U(\mathcal{H})$ such that

$$u'(v_{n+1} \otimes v) = \big(q(F)n + r(F)\big)\, v_n \otimes v,$$

where $q(F), r(F) \in \mathbb{Q}(\alpha, \beta, h, h_I, c_L, c_{LI})\,[F]$, $\deg q = p - 1$, $\deg r = p$. Again, this shows $U_{n-1} \subseteq U_n$. ■

Theorem 7.9 *$V'_{\alpha,\beta,F} \otimes L(\overline{c}, 0, 0)$ is irreducible if and only if $\alpha \notin \mathbb{Z}$. Moreover,*

$$(V'_{0,\beta,F} \otimes L(\overline{c}, 0, 0))/U_0 \cong V(\overline{c}, 1 - \beta, F), \text{ if } \beta \neq 1,$$
$$(V'_{0,1,F} \otimes L(\overline{c}, 0, 0))/U_0 \cong V(\overline{c}, 1, F).$$

Proof. Let $\alpha \notin \mathbb{Z}$. Since $L_{-1} v = 0$ in $L(\overline{c}, 0, 0)$ we have

$$L_{-1}(v_n \otimes v) = -(n + \alpha) v_{n-1} \otimes v$$

which prooves $U_n = U_{n-1}$ for all n, so $V'_{\alpha,\beta,F} \otimes L(\overline{c}, 0, 0)$ is irreducible.

Now let $\alpha = 0$. We may set $p = r = 1$ in the proof of Theorem 5.8 to show reducibility of $U_{-1} = V'_{\alpha,\beta,F} \otimes L(\overline{c}, 0, 0)$. In particular, U_0 is an irreducible submodule. U_{-1}/U_0 is the highest weight module with the highest weight $(\overline{c}, 1 - \beta, F)$ if $\beta \neq 1$, and U_{-2}/U_0 is the highest weight module of the highest weight $(\overline{c}, 1, F)$ if $\beta = 1$. It is left to prove that quotient module is free over $U(\mathcal{H}_-)$. Since U_0 is spanned with vectors $x(v_k \otimes v)$, where $x \in U(\mathcal{H}_- \setminus \{L_{-1}\})$, $k \in \mathbb{Z}_+$, it follows that every vector in U_0 has a component $v_k \otimes xv$ for some $k \geq 0$. Therefore, $y(v_{-1} \otimes v) \notin U_0$ for any $y \in U(\mathcal{H}_-)$, so U_{-1}/U_0 is free, i.e. a Verma module. ■

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